Multivariant system of perspective

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Abstract

The article describes a patented method of displaying objects that secures natural visual perception of the objects displayed without distortions in their scaling relation and the dimensional depth. The author considers a group of linear perspectives with homographic function to display/map 3D visual scenes. The article also describes matrices for projective transformation.

Keywords: Perspective projection, perceptive mapping, group of linear perspectives

1. Introduction

All mainstream Application Program Interfaces (APIs) for 3D Graphics like OpenGL and Direct3D, and corresponding computer hardware (video adapters) still use the projective transformation as it is known in Geometric Optics. Thus they all use The Renaissance Perspective as it was mastered in the time of Leonardo da Vinci (1452-1519). The problem how to display 3D scenes perceived by a human eye has been well-studied by nowadays [1-7]. Knowledge on the visual space’s geometrical structure and cognitive processes of visual perception has resulted in General Theory of Projections, which has been realized in the modern graphic tools [8]. Significant advancement in this field could be made through the method for displaying objects [9], which allows multivariant system of prospective, in other words, a group of linear perspectives with homographic function to display/map 3D visual scenes. The method has been patented and numbered 79 in “100 Russia’s Best Inventions” Chart of Rospatent, the Russian Federal Service for Intellectual Property [10].

2. Method of object imaging (versions)

The method, as named above, corresponds to psychological essence of visual perception and summons a group of perspectives, including the renaissance one and axonometry as the boundary versions of the perceptive system [1].

Let for coordinate system XYZ (Fig. 1a) P is a point, K is a tangent plane, matching coordinate plane XOY and E is a center of projection which is in \( z_0 \) distance from the origin of coordinates. The first way of displaying objects under the method is central projection of the objects’ points, for instance P, to picture plane K (point P'). To perform that one projects P along perpendicular PA to plane K to distance AP, which is a product of initial distance PA (form P to plane K) and compression ratio \( \alpha \) at point P. After P has been transferred to position P', one executes central projection of the point with ray EP to tangent plane K at point P'.

The second way to display an object is to project every object’s point, for instance P, to plane K along ray E’P, which crosses the virtual center of projection E’ located on axis OZ in the distance from plane K, which is a quotient of distance \( z_0 \) and compression ratio \( \alpha \) at point P.
3. **Perspective projection**

So, one has point \( P (x, y, z) \) set in coordinate system \( XYZ \) (Fig. 1a). The space compression ratio is equal to \( \alpha \). \( P_S(x, y, \alpha z) \) is point \( P \) has been shifted. \( E(0, 0, -z_0/\alpha) \) is the center of the projection. \( E'(0, 0, -z_0) \) is the virtual center of the projection. At this, perspective projection of point \( P \) to plane \( XOY \) is expressed as:

\[
\begin{align*}
x' &= \frac{x_0}{\alpha z_0 + x} = \frac{x(z_0/\alpha)}{z + (z_0/\alpha)}, \\
y' &= \frac{y_0}{\alpha z_0 + y} = \frac{y(z_0/\alpha)}{z + (z_0/\alpha)}, \\
z' &= 0,
\end{align*}
\]

where \( (x', y', z') = P' \) are coordinates of \( P \)’s projection. In the expressions for \( x' \) and \( y' \) the first fraction corresponds the first way \((P \rightarrow P_S \rightarrow P' \rightarrow E)\), while the second one corresponds to the second way \((P \rightarrow P' \rightarrow E')\). From (1) one can conclude the images made in the 2 ways are congruent.

The matrix solution for the points’ homogenous coordinates is expressed as:

\[
P' = M' \cdot P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \alpha/z_0 & 1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z \\
1
\end{pmatrix} = \begin{pmatrix}
x \\
y \\
z \\
1
\end{pmatrix} \frac{1}{(\alpha/z_0)z + 1} \begin{pmatrix}
x \\
y \\
z \\
1
\end{pmatrix},
\]

where \( M' \) is projection matrix; "\( \Rightarrow \)" stands for an operation of “perspective” subtraction. The transform is singular, since one loses coordinate \( z' \) and the dimensional depth.

4. **Perceptive mapping**

The perceptive mapping (2) forms a two-dimensional image. Here one considers the mapping that transforms 3D object space into 3D image space.

Figure 1b demonstrates the second way to display objects with the certain changes and additions inserted. The rectangular coordinate system has been transferred along axis \( OZ \) in such a way that origin of the coordinates \( O \) matches center of projection \( E \) (an eye of an observer). Picture plane \( K \) is in the same \( z_0 \) distance from the center of projection.

![Fig. 1](https://via.placeholder.com/150)
Perspective projection $P'(x', y', z')$ of point $P(x, y, z)$ determines abscissa $x''=x'$ and ordinate $y''=y'$ of certain point $P''(x'', y'', z'')$ of the image space. Applicate $z''$ of this point is determined taking into account that an eye sees point $P''$ at the same angle $\theta$, as it sees point $P$. It means that:

$$\frac{x'}{x} = \frac{x''}{x} = \frac{y'}{y} = \frac{y''}{y} = \frac{z''}{z} = \frac{z_0/\alpha}{z - z_0 + z_0/\alpha} = \frac{1}{(\alpha/z_0)z + 1 - \alpha} = f(z),$$

where $f(z)$ is a function of perceptive mapping. Thus, $P''=P f(z)$ is a way to project object space to image space.

The matrix solution for the points’ homogenous coordinates is expressed as:

$$P'' = M^n P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 - \alpha \\ \alpha/z_0 & 1 - \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x'' \\ y'' \\ z'' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ (\alpha/z_0)z + 1 - \alpha \\ y \\ (\alpha/z_0)z + 1 - \alpha \\ z \\ (\alpha/z_0)z + 1 - \alpha \\ 1 \end{pmatrix},$$

where $M^n$ is a matrix of projective mapping.

5. **Some features of perceptive mapping**

1. The mapping has four stationary planes. The three of them are coordinate planes XOZ, XOY, YOZ. For instance, points $P(0, y, z)$ having been mapped (4) are transferring in plane YOZ (Fig. 1b) and so on. The fourth stationary plane is picture plane $K$ that is point-invariant.

Points $P(x, y, z_0)$ that belong to the tangent plane are invariant to mapping. $P''(0, 0, 0) = P(0, 0, 0)$. Thus, after the mapping takes place, the direction of axis of vision $OZ$, upward direction $OY$ and rightward direction $OX$ have remained unchanged.

2. At uniform space compression of objects along axis $OZ$ to tangent plane $K$ with compression ratio $\alpha$ the mapping forms a group of linear perspectives. In this case the straight line connecting two points in object space is transforming into a straight line connecting two points mapped in image space. Uniform space compression is being secured by specified value of compression ration $\alpha$ for each of the perspectives.

3. When $\alpha=0$ it secures identical mapping $P''=P$. That makes different types of parallel projection possible, including axonometry.

4. When $0<\alpha<1$, 3D perceptive perspective occurs, which takes into account the personal constants of an observer through values $\alpha$ and $z_0$. It is easy to demonstrate that $\alpha=z_0/(z_0+d)$, where $d$ is a hyperfocal distance in a reduced eye model. Here one follows Emmert’s Law: If the size of the retinal image of an object remains constant, perceived size of the object is directly proportional to perceived distance to this object. The value of the retinal image has been constant at constant angle $\theta$ (Fig. 1b). If $y''$ is a perceived size of an object, than they can be expressed by direct ratio $y''=z'' \tan \theta$.

5. When $\alpha=1$ one can observe Renaissance perspective $P''(xz_0/z, yz_0/z, z_0)$ with specified coordinate $z''=z_0$. Thus in case of Renaissance perspective one has degenerate mapping and a two-dimensional image.

6. When $\alpha>1$, the object space is not being compressed but spread. The objects displayed in the far zone are shrinking in their size in comparison to the size of objects in Renaissance perspective, while those in the near zone are getting bigger. This is a perspective with reverse depth because increasing distance $z$ in object space results in decreasing distance $z''$ in image space. As a whole, the image is getting more wide-angle.
7. When $\alpha<0$ one observes a reverse perspective, when remote objects are getting bigger in image space, while close ones are getting smaller.

Figure 2 demonstrates the relative depths and sizes of objects as they have been calculated according to the formula (4) for different systems of linear perspectives. The curves are crossing at the point in distance $z_0=2$ m, where the point-invariant tangent plane is located.

![Graph showing sizes and depths of objects at different values of the compressing ratio.](image)

Fig. 2. The sizes and depths of objects at different values of the compressing ratio:
1) $\alpha=-0.4$; 2) $\alpha=0$; 3) $\alpha=0.4$; 4) $\alpha=1$; 5) $\alpha=1.4$.

Figure 2 shows that the axonometric (2) and perceptive (3) perspectives can be applied without any limitations, including 3D images. The Renaissance (4) and wide-angle (5) perspectives can be effective in the far zone and in the space beyond the tangent plane, while the reverse perspective (1) – in the space before the tangent plane. Perceptive system 3 [1, 4-6, 8] should be regarded as being the most acceptable one for a human eye.

6. Conclusion

The article describes a patented method (in two variants) of displaying objects. Its application secures natural visual perception of an object space without distortions in their scaling relation and the dimensional depth. At nonuniform space compression it allows one to project 2D images using the degenerated mapping matrix. Using the method the author has built perceptive 3D image, which determines a group of linear perspectives and include axonometric, perceptive, Renaissance, wide-angle and reverse ones. The article describes some of the features of the perspectives in the group.
References

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