Control of the parameter values on the basis of measurement results and associated uncertainties

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Abstract

Conformity assessment of scalar quantity value \( a \) to the tolerance interval \([-A, A]\) is considered. The decision is based on the measurement results and associated uncertainty \( [\hat{a}, u(a)] \). Some empirical approaches applied in practice [1-3] are discussed and commented. The general approach based on Bayesian decision theory is considered. The conventional risks are calculated for Gaussian probability density of the quantity of interest and for a particular loss function. Determination of an acceptance interval for measured values is discussed.

Keywords: Measurement uncertainty, decision rule, tolerance interval, acceptance interval

1. Introduction

The problem of parameter value control in terms of a result obtained in measuring this parameter, is of great importance for practical application. In normative documents, e.g. in [1-3], possible situations and approaches to the way how to take into account the uncertainty in making a decision with regard to the value of a controlled parameter, are considered.

There are situations when an uncertainty interval is completely inside the region of allowed values or outside of it. In this case, the decision-making is trivial. An unambiguous conclusion is impossible, when a part of the uncertainty interval goes beyond the region boundaries.

Obviously, in this case, while taking a decision relative to the parameter position within the region of allowed values, it is impossible to avoid errors of the first or second order. To solve the problem, some additional information about potential consequences in case of such or other errors, is needed.

With a view to make the situation more determinate, we will consider it, when the acceptance of an erroneous decision about the position of a controlled parameter within the allowed boundaries, results in more significant losses than the erroneous conclusion that the parameter is beyond the allowed boundaries.

Below, a number of different practical approaches to formation of a rule for making a decision (RMD) will be considered. Moreover, a loss function of a specific types will be suggested, and the Bayesian theory of making decisions will be applied.

2. Formalized definition of the problem

Formalized description of the problem includes the following parameters:
- Field of possible values of a controlled parameter. In case of consideration it is the whole number axis \( R \). Sometimes in addition, a prior density of the \( p(a) \) parameter distribution is assumed to be the known one. It describes the variability of some technology process. In case of a random choice of a sample when the control is
performed, its value is distributed by $p(a)$. Moreover, the region of allowed values of the controlled parameter is given: $a \in A$.

- RMD that determines how the conclusion about the conformity with the requirements, based on the measurement result $[\hat{a}, u(a)]$, is made, i.e. about the acceptance of the zero hypothesis $H_0 : a \in A$, or about the acceptance of the alternative hypothesis $H_1 : a \notin A$. Thus, for the problem considered, there are only two possible solutions based on the measurement result: to accept $H_0$ or $H_1$. The RMD is the reflection function from a set of measurement results in a set of possible decisions $D = \{d_0, d_1\}$:

$$d(\hat{x}) \rightarrow \begin{cases} d_0 = \text{take } H_0 \\ d_1 = \text{take } H_1 \end{cases}.$$ 

In practice, empirical rules of decision acceptance, based on agreement between a producer and customer, is often used, e.g. “if $\hat{a} \in A$ and $u(a) \leq u_0$, then a value of the controlled parameter is within the region of allowed values, i.e. $H_0$ is accepted”. A general approach to determination of an optimal solution requires the loss function assignment.

- The loss function is given on a set $R \times D$ and determines losses at acceptance of an erroneous decision $L(a, d(\hat{a}))$.

3. Approaches to formation of the RMD

The result of measurement is presented by one of the ways given below:

- $[\hat{a}, u(a)]$, i.e. by estimate of the measurand and associated uncertainty;
- $pdf(a)$, i.e. by the probability density function of possible values of the measurand.

Provided all available measurement information is present in a pair $[\hat{a}, u(a)]$, then the Gaussian probability density is assigned to the measurand values:

$$pdf(a|\hat{a}, u(a)) = \frac{1}{\sqrt{2\pi u(a)}} \exp \left\{ -\frac{(a - \hat{a})^2}{2u^2(a)} \right\}.$$ 

If it is supposed that a prior distribution of the controlled parameter is available, then the posteriori distribution density, according to the Bayes’ theorem, will have the following form:

$$pdf(a|\hat{a}, u(a)) \propto \frac{\exp \left\{ -\frac{(a - \hat{a})^2}{2u^2(a)} \right\}}{\sqrt{2\pi u(a)}} \cdot p(a).$$ 

The availability of the probability density permits to calculate conditional probabilities of a zero and alternative hypothesis $H_0$ and $H_1$:

$$P(H_0|\hat{a}) = \frac{1}{\sqrt{2\pi u(a)}} \int_{a \in A} \exp \left\{ -\frac{(a - \hat{a})^2}{2u^2(a)} \right\} da,$$

$$P(H_1|\hat{a}) = \frac{1}{\sqrt{2\pi u(a)}} \int_{a \notin A} \exp \left\{ -\frac{(a - \hat{a})^2}{2u^2(a)} \right\} da.$$ 

The RMD can be formulated reasoning from the requirements for one of these two conditional probabilities, for example: “if $P(H_0|\hat{a}) \geq 0.7$, then a decision is taken that the controlled parameter value is within the region of allowed values, i.e. the accepted value is
The Bayesian approach to taking a decision in the case given, is the following: they accept the hypothesis for which the conditional probability is greater.

In practice, the region of accepting the zero hypothesis, i.e. some interval for the measurement results, is established. Suppose, that the region of allowed values of the controlled parameter is the interval. Not detracting from the generality, it is possible to consider that it is the interval symmetrical relative to zero \([-A, +A]\).

For the latter, it is enough to assume that the control is realized with regard to deviations of the parameter from some established level.

In the case, when consequences of the erroneous acceptance of \(H_0\) are the most undesirable ones, the area of the \(H_0\) acceptance has the form:

\[-A + m \leq \hat{a} \leq A - m.\]

Let us determine \(m\) from the condition:

\[P(H_0 \mid -A + m \leq \hat{a} \leq A - m) > P_0.\]

A minimal value of the conditional probability is achieved at the ends of the interval. For the sake of definiteness, the right-hand end \(\hat{a} = A - m\) will be examined. Then the condition for \(m\) will have the form:

\[P(H_0 \mid \hat{a} = A - m) = P_0.\]

From it follows:

\[P_0 = P(H_0 \mid \hat{a} = A - m) = \frac{1}{\sqrt{2\pi u(a)}} \int_{-A}^{A} \exp \left\{ -\frac{(a - A + m)^2}{2u^2(a)} \right\} da.\]

After substitution of the variables

\[x = \frac{a - A + m}{u(a)}\]

we get:

\[P_0 = \frac{1}{\sqrt{2\pi}} \int_{\frac{m}{u(a)}}^{\frac{m}{u(a)}} \exp \left\{ -\frac{x^2}{2} \right\} dx \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{m}{u(a)}} \exp \left\{ -\frac{x^2}{2} \right\} dx.\]

Thus, the ratio \(\frac{m}{u(a)}\) is the quantile of the normal distribution, and for the given value \(P_0\) the ratio \(\frac{m}{u(a)} = z_{P_0}\) from appropriate tables is determined. Therefore, the expression for \(m\) has the form \(m = z_{P_0} \times u(a)\).

Hence, the more is the uncertainty of the measurement result, the larger is the band \(m\). At the approach of such a kind, the zero hypothesis will be erroneously refuted with a great probability, when the measurement result insignificantly exceeds \(A - z_{P_0} \times u(a)\). The approach set out does not take into account the circumstance that, as a rule, unwanted consequences of the controlled parameter overrun beyond the permissible limit, are proportional to the value of this deviation. Therefore, it is natural to suggest the following approach to the RMD formation: to apply limits to average losses.

Below a particular case of establishing the primitive linear loss function is considered:

\[R(\hat{a}) = \frac{1}{\sqrt{2\pi u(a)}} \int_{-A}^{A} (a - A) \exp \left\{ -\frac{(a - \hat{a})^2}{2u^2(a)} \right\} da + \frac{1}{\sqrt{2\pi u(a)}} \int_{-\infty}^{\frac{m}{u(a)}} (a - A) \exp \left\{ -\frac{(a - \hat{a})^2}{2u^2(a)} \right\} da \leq C.\]
For explicitness, suppose that the measurement result is near the upper boundary of the allowed value region, then:

\[
R(\hat{a}) \approx \frac{1}{\sqrt{2\pi u(a)}} \int_{-\infty}^{\hat{a}} (a-A) \exp \left\{ -\frac{(a-\hat{a})^2}{2u^2(a)} \right\} da.
\]

By substituting \( x = \frac{a-\hat{a}}{u(a)} \) and integrating in parts, we get:

\[
R(\hat{a}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\hat{a}} (x \cdot u(a) + \hat{a} - A) \exp \left\{ -\frac{x^2}{2} \right\} dx = \frac{1}{u(a)} \int_{-\infty}^{\hat{a}} \exp \left\{ -\frac{(A-\hat{a})^2}{2u^2(a)} \right\} + \frac{\hat{a} - A}{u(a)} \Phi \left( \frac{\hat{a} - A}{u(a)} \right).
\]

The expression for \( R(\hat{a}) \) depends on a standard uncertainty \( u(a) \) of the measurement and closeness of the measurement result to an extremely allowed value of the controlled parameter \( A - \hat{a} \), as well as on their ratio. The maximum value is obtained when \( \hat{a} = A \) and completely set by the measurement uncertainty. Further, it is possible to achieve a significant decrease of losses due to a choice of the zero hypothesis acceptance region with a some margin: \( \frac{A - \hat{a}}{u(a)} > \alpha \).

The general approach to the problem solution is based on setting the loss function which is more close to a metrological practice. Let us consider that the losses due to an admissible spillover of the controlled parameter beyond the boundary, are proportional to the value of this deviation. As to losses due to quality control and rejection of suitable objects as the defective ones, they are expressed in the form of some constant. For the description of this situation, the following loss function can be suggested:

\[
L(a, d(\hat{a})) = \begin{cases} c_0 \mid a - A \mid^\rho, & a \notin A, d_0 \\ 0, & a \notin A, d_1 \\ c_1, & a \in A, d_1 \end{cases}
\]

A conditional risk at the obtained measurement result is calculated as:

\[
R(d_0|\hat{a}) = \frac{c_0}{\sqrt{2\pi u(a)}} \int_{A}^{\hat{a}} (|a| - A)^\rho \exp \left\{ -\frac{(a-\hat{a})^2}{2u^2(a)} \right\} p(a) da,
\]

\[
R(d_1|\hat{a}) = \frac{c_1}{\sqrt{2\pi u(a)}} \int_{-\infty}^{\hat{a}} \exp \left\{ -\frac{(a-\hat{a})^2}{2u^2(a)} \right\} p(a) da.
\]

In accordance with the Bayes’ theory, the decision that provides a less risk is accepted. To illustrate the approach, we will consider the case when \( \rho = 1 \), and the prior distribution function of the controlled parameter is Gaussian:

\[
p(a) = \frac{1}{\sqrt{2\pi \sigma}} \exp \left\{ -\frac{a^2}{2\sigma^2} \right\}.
\]

In this case the aposterior pdf of the controlled parameter has the form:

\[
p(a|\hat{a}) = \frac{1}{\sqrt{2\pi \sigma_w}} \exp \left\{ -\frac{(a-\hat{a}_w)^2}{2\sigma_w^2} \right\}, \quad \hat{a}_w = \frac{1}{u^2(a)} + \frac{1}{\sigma^2}, \quad \sigma_w = \left( \frac{1}{u^2(a)} + \frac{1}{\sigma^2} \right)^{\frac{1}{2}}.
\]
\[ R(d_d | \hat{a}) = c_0 \sigma_w \left\{ \exp \left\{ \frac{(A - \hat{a}_w)^2}{2\sigma_w^2} \right\} + \exp \left\{ \frac{(A + \hat{a}_w)^2}{2\sigma_w^2} \right\} \right\} - \sqrt{2\pi} + \hat{a}_w - A \frac{\hat{a}_w - A}{\sigma_w} + A \frac{A + \hat{a}_w}{\sigma_w} \cdot \Phi \left( \frac{\hat{a}_w - A}{\sigma_w} \right) - \Phi \left( -\frac{A + \hat{a}_w}{\sigma_w} \right) \]

\[ R(d_i | \hat{a}) = c_i \left( \Phi \left( \frac{A - a_w}{\sigma_w} \right) - \Phi \left( -\frac{A + a_w}{\sigma_w} \right) \right). \]

In accordance with the Bayes’ theory, the decision is accepted that provides a less conditional risk at an obtained measurement result. At the given \( c_0, c_1 \) loss costs and relation between the width of a tolerance between the measurement uncertainty and dispersion of the controlled process, the last two expressions permit to calculate the region of the zero hypothesis acceptance, i.e. the interval of the values \( \hat{a}_w \) for which the conditional risk in accepting the decision \( d_0 \) appears to be less.

4. Conclusion
The problem of the parameter control on the basis of its measurement results and associated uncertainties has been considered. For the Gaussian distribution of measurand values, the explicit expression of conditional risks has been obtained. The recommendations on formation the regions of permissible results of controlled parameter measurements have been given.

References